

Differentiability of Distance Function and The Proximinal Condition implying Convexity

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Abstract

A necessary and sufficient condition for the differentiability of the distance function generated by an almost proximinal closed set has been given for normed linear spaces with locally uniformly convex and differentiable norm. We prove that the proximinal condition of Giles (Proc. Amer. Math. Soc., 104, No. 2, 1988, 458-464) is true for almost sun. In such spaces if the proximinal condition is satisfied and the distance function is uniformly differentiable on a dense set then it will result in the differentiability on all off the set (generating the distance function). The proximinal condition ensures about the convexity of almost sun in some spaces under a differentiability condition of the distance function. A necessary and sufficient condition is obtained for the convexity of Chebyshev sets in Banach spaces with rotund dual.

Keywords. Distance function, Proximinal set, Differentiability, Generalized subdifferential, Almost sun, Chebyshev set.

1 Introduction

Let X be a real normed linear space. For a nonempty closed set K in X , we define its distance function d_K on X by

$$d_K(x) = \inf \{ \|x - k\| : k \in K \}.$$

This function is not necessarily every where differentiable but it is (globally) Lipschitz, with the Lipschitz constant equal to 1. The metric projection of x into K is

$$P_K(x) = \{k \in K : \|x - k\| = d_K(x)\}.$$

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The set K is called proximal (Chebyshev) if for every $x \in X \setminus K$, $P_K(x)$ is nonempty (singleton). K will be called almost proximal if $P_K(x)$ is nonempty for a dense set of $x \in X \setminus K$. A proximal set K in a normed linear space X is a sun if for every $x \in X \setminus K$ with a closest point $p(x) \in K$, points $x + t\vec{x}$ have $p(x)$ as a closest point for all $t \geq 0$, where \vec{x} is a unit vector in the direction of $x - p(x)$. An almost proximal set K will be called almost sun if for a dense set of $x \in X \setminus K$ with a closest point $p(x) \in K$, points $x + t\vec{x}$ also have $p(x)$ as a closest point for all $t \geq 0$.

Dutta [4] has deduced that if the norm on X is locally uniformly convex (LUR) and (Fréchet) smooth. Then the (Fréchet) smoothness of the distance function d_K generated by an almost proximal set K is generic on $X \setminus K$. Further if norms on X and X^* are LUR, then he characterized the convexity of Chebyshev sets in terms of the Clarke generalized subdifferential of the distance function. His technique is based on the observation of the denseness of the set $E'(K)$, where $E'(K)$ denotes the set of points in $X \setminus K$ for which every minimizing sequence in K converges to a unique nearest point. A sufficient condition for $E'(K)$ to be dense is the local uniform convexity of the norm on X . We will use this result to improve (in some sense new) results of Giles [6].

In a normed linear space X , Giles [6] assumed a proximal condition on a nonempty closed set K , which has the property that for some $r > 0$ there exists a set of points $x_o \in X \setminus K$ which have closest points $p(x_o) \in K$ with $d_K(x_o) > r$ such that the set of points $x_o - r\vec{x}_o$ is dense in $X \setminus K$. It has been shown that if the norm has sufficiently strong differentiability properties, then the distance function d_K generated by K has similar differentiability properties and it follows that, in some spaces, K is convex.

It is well known that in a smooth finite-dimensional normed linear space every Chebyshev set is convex and the metric projection is continuous on $X \setminus K$ and this fact is used in the proof. So it is natural to consider the continuity of the metric projection while proving the convexity of Chebyshev sets in smooth infinite-dimensional spaces. The best known result is due to Vlasov [7]: in a Banach space with rotund dual, Chebyshev sets with continuous metric projection are convex. A close look of Vlasov's proof shows that the continuity of the metric projection has been used only to establish a differentiability condition of the distance function generated by the set. In terms of a differentiability condition on the distance function, Vlasov's Theorem can be stated as follows.

Proposition 1.1. (Borwein et al. [1], THEOREMS 14-18).

In a Banach space X with rotund dual X^ , a nonempty closed set K is convex if its distance function d_K satisfies*

$$\limsup_{\|y\| \rightarrow 0} \frac{d_K(x+y) - d_K(x)}{\|y\|} = 1 \quad \text{for all } x \in X \setminus K.$$

In particular, this differentiability condition is satisfied if d_K is smooth and $\|d'_K(x)\| = 1$ or if d_K is Fréchet smooth for all $x \in X \setminus K$.

It is easily seen that in any normed linear space X , if every point $x \in X \setminus K$ is an interior point of an interval with end points $x_o \in X \setminus K$ and closest point $p(x_o) \in K$, then $p(x) = p(x_o)$

and the differentiability condition of Proposition 1.1 will be satisfied. For the convexity of K in a normed linear space X , it is a necessary condition that the distance function d_K satisfies the differentiability condition for all $x \in X \setminus K$. We will use Vlasov's Theorem in the form of Proposition 1.1 to establish convexity results. Fitzpatrick [5] observed a close connection between continuity of the metric projection and differentiability of the distance function, and a differentiability condition on the distance function implies convexity of Chebyshev sets.

To make the paper self contained, we reproduce some definitions and known results given as follows.

A function $h : X \rightarrow \mathbb{R}$ is said to be *Gâteaux differentiable* or *smooth* at $x \in X$ if there exists a continuous linear functional $h'(x) \in X^*$, called the Gâteaux derivative of h , such that for given $\epsilon > 0$ and $y \in X$ with $\|y\| = 1$ there exists a $\delta(\epsilon, x, y) > 0$ such that

$$\left| \frac{h(x + ty) - h(x)}{t} - h'(x)(y) \right| < \epsilon \text{ when } 0 < |t| < \delta. \quad (1)$$

The function h is said to be *Fréchet smooth* at x if there exists a $\delta(\epsilon, x) > 0$ such that inequality (1) holds for all $y \in X$ with $\|y\| = 1$.

The function h is said to be *uniformly smooth* on a set D if there exists a $\delta(\epsilon, y) > 0$ such that inequality (1) holds for all $x \in D$, and is said to be *uniformly Fréchet smooth* on a set D if there exists a $\delta(\epsilon) > 0$ such that inequality (1) holds for all $x \in D$ and for all $y \in X$ with $\|y\| = 1$.

The space X is said to be *smooth* (*Fréchet smooth*) at $x \neq 0$ if the norm is smooth (Fréchet smooth) at $x \neq 0$. We say that X has *uniformly smooth* (*uniformly Fréchet smooth*) norm if the norm is uniformly smooth (uniformly Fréchet smooth) on the unit sphere $\{x \in X : \|x\| = 1\}$.

Let $h : X \rightarrow \mathbb{R}$ be a locally Lipschitz function. The *Clarke generalized directional derivative* of h at a point x and in the direction $y \in X$, denoted by $h^\circ(x; y)$, is given by:

$$h^\circ(x; y) = \limsup_{z \rightarrow x, t \downarrow 0} \frac{h(z + ty) - h(z)}{t}$$

and the *Clarke generalized subdifferential* of h at x is given by

$$\partial h(x) = \{f \in X^* : h^\circ(x; y) \geq f(y), \forall y \in X\}.$$

2 Differentiability and The Proximinal Condition

We denote by $E(K)$, the set of all points in $X \setminus K$ which has nearest points in K and $E'(K)$ to be the set of $x \in E(K)$ such that every minimizing sequence in K for x converges to a unique nearest point of x . We denote by f_x , a subgradient of the norm at $x \in X$, then subdifferential $\partial \|x\|$, the set of all subgradients of the norm at $x \in X$, is given by

$$\partial \|x\| = \{f_x \in X^* : f_x(x) = \|x\| \text{ and } \|f_x\| = 1\}$$

Note that $\partial \left\| \frac{x}{\|x\|} \right\| = \partial \|x\|$ for $x \neq 0$. Clearly, if norm is smooth at $x \neq 0$ then $\partial \|x\|$ is singleton. In this case the single subgradient f_x becomes Gâteaux derivative.

The following two lemmas play very crucial role in establishing our results.

Lemma 2.1. (Borwein and Giles [2], LEMMA 1) *For any $z \in E(K)$ and every $p(z) \in P_K(z)$, there exists an $f_z \in \partial \|z - p(z)\|$ such that $f_z \in \partial d_K(z)$.*

Lemma 2.2. (Dutta [4], LEMMA 1) *For any $z \in E'(K)$ we have*

$$\partial d_K(z) \subseteq \partial \|z - P_K(z)\|.$$

Equality holds if norm on X is smooth at $z - P_K(z)$. Moreover norm on X is Fréchet smooth at $z - P_K(z)$, then d_K is Fréchet smooth at z .

For more detailed explanation of generalized subdifferential, see Clarke [3].

Remark 2.1. *Indeed, smoothness of d_K in Lemma 2.1 is strict (see [3], Proposition 2.2.4). For a locally Lipschitz function h , the strict differentiability is equivalent to being singleton of ∂h .*

Now, we establish the following result, which would serve as a part of next result of this paper.

Proposition 2.1. *Let X be a smooth normed linear space. Let K be a nonempty closed set with the set $E'(K)$ dense in $X \setminus K$. Suppose $x \in X \setminus K$ is such that for every $y_n \in E'(K)$ with $y_n \rightarrow x$ the sequence $\{g_{\tilde{y}_n}\}$ is w^* -convergent. Then the distance function d_K generated by K is strictly smooth at x .*

Proof. To prove that d_K is strictly smooth at $x \in X \setminus K$, it suffices to show that $\partial d_K(x)$ is singleton.

Let $y_n \in E'(K)$ be any sequence such that $y_n \rightarrow x$. By definition of upper limit, for each $n \in \mathbb{N}$ there exists $z_n \in X \setminus K$ and $t_n > 0$ such that

$$\|z_n - y_n\| + t_n < \frac{1}{n}, \text{ and}$$

$$\begin{aligned} d_K^\circ(y_n; y) - \frac{1}{n} &\leq \frac{d_K(z_n + t_n y) - d_K(z_n)}{t_n} \\ \text{hence, } \limsup_{n \rightarrow \infty} d_K^\circ(y_n; y) &\leq \limsup_{z \rightarrow x, t \downarrow 0} \frac{d_K(z + ty) - d_K(z)}{t} \\ &= d_K^\circ(x; y). \end{aligned}$$

Since $y_n \in E'(K)$ with $y_n \rightarrow x$, so by Lemma 2.2, $\partial d_K(y_n) = \partial \|y_n - P_K(y_n)\|$ is singleton, we have $d_K^\circ(y_n; y) = d'_K(y_n)(y) = g_n(y)$, where $g_n(y_n - P_K(y_n)) = \|y_n - P_K(y_n)\|$, that is $g_n = g_{\tilde{y}_n}$, with $y_n \rightarrow x$ and $y_n \in E'(K)$, hence w^* -convergent. Let $g_{\tilde{y}_n} \rightarrow g$ in w^* -topology, by w^* -upper semicontinuity of ∂d_K , we must have $g \in \partial d_K(x)$.

Thus, for all $y \in X$ with $\|y\| = 1$ and for every sequence $y_n \in E'(K)$ with $y_n \rightarrow x$, we have $\lim_{n \rightarrow \infty} d_K^\circ(y_n; y)$ exists in $\partial d_K(x)(y)$, so linear in y and

$$\lim_{n \rightarrow \infty} d_K^\circ(y_n; y) \leq d_K^\circ(x; y).$$

Now, we prove that the reverse of the last inequality holds for some $y_n \in E'(K)$ with $y_n \rightarrow x$. Which proves that $d_K^\circ(x; y)$ is linear in y , and it follows that $\partial d_K(x)$ is singleton.

For, let $y \in X$ with $\|y\| = 1$, then by definition of $d_K^\circ(x; y)$, corresponding to each n there exists $z_n \in X \setminus K$ and $t_n > 0$ such that

$$\|z_n - x\| + t_n < \frac{1}{n}, \text{ and}$$

we have

$$d_K^\circ(x; y) - \frac{1}{n} \leq \frac{d_K(z_n + t_n y) - d_K(z_n)}{t_n}.$$

Since $E'(K)$ is dense in $X \setminus K$, choose $y_n \in E'(K)$ such that $\|z_n + t_n y - y_n\| < t_n^2$. Then $d_K(z_n + t_n y) \leq d_K(y_n) + t_n^2$ and $d_K(z_n) \geq d_K(y_n - t_n y) - t_n^2$. Thus for sufficiently large n , we have

$$\begin{aligned} d_K^\circ(x; y) - \frac{1}{n} &\leq \frac{d_K(y_n) - d_K(y_n - t_n y)}{t_n} + 2t_n \\ &= \frac{d_K((y_n - t_n y) + t_n y) - d_K(y_n - t_n y)}{t_n} + 2t_n \\ &\leq d_K^\circ(y_n; y) + \frac{1}{n} + 2t_n. \end{aligned}$$

Thus

$$d_K^\circ(x; y) \leq \lim_{n \rightarrow \infty} d_K^\circ(y_n; y).$$

This completes the proof of Proposition.

Remark 2.2. *It may be noted that merely density condition is not sufficient for the conclusion of Proposition 2.1. Let us consider $K = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$, the unit sphere in smooth space $X = \mathbb{R}^2$, then $E'(K) = X \setminus (K \cup \{0\})$. Put $x = 0$ then we can find sequences $\{y_n\}$ of non zero vectors such that the sequences converge to zero but $\{g_{\tilde{y}_n}\}$ are not convergent, so d_K is not strictly smooth at $x = 0$. In fact the subdifferential of d_K is the closed unit ball and it is easy to see that d_K is even not smooth at $x = 0$.*

□

Before proceed further to establish our main results, note that if X is smooth then for any $z_n \in E(K)$ and every $p(z_n) \in P_K(z_n)$, the subdifferential $\partial \|z_n - p(z_n)\|$ is singleton which depends on z_n and $p(z_n)$ both. Since $P_K(z_n)$ is set valued, so $f_{\tilde{z}_n} \in \partial \|z_n - p(z_n)\|$ need not be unique corresponding to given $z_n \in E(K)$. Indeed, every sequence $\{z_n\}$ in $E(K)$ determines (possibly uncountably) many sequences $\{f_{\tilde{z}_n}\}$. Hence when we say that $\{f_{\tilde{z}_n}\}$ is $(w^*$ - or norm) convergent for every sequence $\{z_n\}$ in $E(K)$ with $z_n \rightarrow x$, we mean that for every $p(z_n) \in P_K(z_n)$ the sequence $\{f_{\tilde{z}_n}\}$ where $f_{\tilde{z}_n} \in \partial \|z_n - p(z_n)\|$, obtained in this way, is $(w^*$ - or norm) convergent. It may be noted that they need not converge to the same $(w^*$ - or norm)

limit. For more details see Borwein et al. [1] Corollary 9 and Giles [6].

The following Proposition provides a necessary and sufficient condition for differentiability of the distance function if $E'(K)$ is dense in $X \setminus K$. We do not assume the uniform differentiability conditions as in Giles [6], Proposition 2. Hence, we hope, our result advances that of Giles [6].

Proposition 2.2. *Let X be a normed linear space with smooth (Fréchet smooth) norm and K be a nonempty closed set with the set $E'(K)$ dense in $X \setminus K$. Then the distance function d_K generated by K is strictly smooth (and Fréchet smooth) at $x \in X \setminus K$ if and only if $\{f_{z_n}\}$ is w^* -convergent (norm convergent) for every sequence $\{z_n\}$ in $E(K)$ with $z_n \rightarrow x$.*

Proof. **First** we consider the case when norm is smooth. Suppose that for every sequence $\{z_n\}$ in $E(K)$ with $z_n \rightarrow x$, the sequence $\{f_{z_n}\}$ is w^* -convergent. Since $E(K)$ contains $E'(K)$, so it follows from Proposition 2.1 that d_K is strictly smooth at x .

Conversely, suppose that d_K is strictly smooth at x . Let $f_{z_n} \in \partial \|z_n - p(z_n)\|$, since norm is smooth so by Lemma 2.1, $f_{z_n} \in \partial d_K(z_n)$. Let f be w^* -cluster point of f_{z_n} , by upper semi-continuity of ∂d_K , we have $f \in \partial d_K(x)$, but d_K is strictly smooth at x , hence the sequence $\{f_{z_n}\}$ is w^* -convergent to $d'_K(x)$.

We **next** consider the case when the norm is Fréchet smooth. Suppose that for every sequence $\{z_n\}$ in $E(K)$ with $z_n \rightarrow x$, the sequence f_{z_n} is norm convergent (so w^* -convergent to $d'_K(x)$). Then d_K is strictly smooth at x . It remains to prove the Fréchet smoothness only.

Since d_K is smooth at x , so for any $t_n \rightarrow 0$ and any $y \in X$ with $\|y\| = 1$,

$$\lim_{n \rightarrow \infty} \frac{d_K(x + t_n y) - d_K(x)}{t_n} = d'_K(x)(y) = \lim_{n \rightarrow \infty} f_{z_n}(y)$$

But $f_{z_n} \rightarrow d'_K(x)$ in norm, so the last limit is uniform over $\|y\| = 1$. Hence d_K is Fréchet smooth at x .

Finally, suppose that d_K is strictly smooth and Fréchet smooth at $x \in X \setminus K$. Then we prove that f_{z_n} is norm convergent to $d'_K(x)$ for every sequence $\{z_n\}$ in $E(K)$ with $z_n \rightarrow x$.

First we prove that f_{y_n} is norm convergent to $d'_K(x)$ for every sequence $\{y_n\}$ in $E'(K)$ with $y_n \rightarrow x$. Since norm is Fréchet smooth, so by Lemma 2.2 d_K is Fréchet smooth at each $y_n \in E'(K)$ and by assumption d_K is Fréchet smooth at $x \in X \setminus K$.

So, for given $\epsilon > 0$ there exists a $\delta_1(\epsilon, y_n) > 0$ and $\delta_2(\epsilon, x) > 0$ such that for all $y \in X$ with $\|y\| = 1$, we have

$$\left| \frac{d_K(y_n + ty) - d_K(y_n)}{t} - f_{y_n}(y) \right| < \epsilon \quad \text{for all } 0 < |t| < \delta_1.$$

and

$$\left| \frac{d_K(x + ty) - d_K(x)}{t} - d'_K(x)(y) \right| < \epsilon \quad \text{for all } 0 < |t| < \delta_2.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$, then for any $y \in X$ with $\|y\| = 1$, we have

$$\begin{aligned}
|f_{y_n}^{\rightarrow}(y) - d'_K(x)(y)| &\leq \left| \frac{d_K(y_n + ty) - d_K(y_n)}{t} - f_{y_n}^{\rightarrow}(y) \right| \\
&+ \left| \frac{d_K(y_n + ty) - d_K(y_n)}{t} - \frac{d_K(x + ty) - d_K(x)}{t} \right| \\
&+ \left| \frac{d_K(x + ty) - d_K(x)}{t} - d'_K(x)(y) \right| \\
&< 2\epsilon + \frac{4}{\delta} \|y_n - x\| \quad \text{for all } \frac{\delta}{2} < |t| < \delta \\
&< 6\epsilon \quad \text{for all } \|y_n - x\| < \epsilon\delta.
\end{aligned}$$

Since $y_n \rightarrow x$, hence the sequence $f_{y_n}^{\rightarrow}(y)$ is uniformly convergent to $d'_K(x)(y)$ over $\|y\| = 1$, that is $f_{y_n}^{\rightarrow}$ is norm convergent to $d'_K(x)$.

Since

$$|f_{z_n}^{\rightarrow}(y) - d'_K(x)(y)| \leq |f_{z_n}^{\rightarrow}(y) - f_{y_n}^{\rightarrow}(y)| + |f_{y_n}^{\rightarrow}(y) - d'_K(x)(y)|$$

hence to complete the proof of the result it is enough to show that for every sequence $\{z_n\}$ in $E(K)$ with $z_n \rightarrow x$ there is some sequence $\{y_n\}$ in $E'(K)$ with $y_n \rightarrow x$, such that $|f_{z_n}^{\rightarrow}(y) - f_{y_n}^{\rightarrow}(y)|$ converges to zero uniformly over $\|y\| = 1$.

Suppose there exists a sequence $\{z_n\}$ in $E(K)$ with $z_n \rightarrow x$ such that $f_{z_n}^{\rightarrow}$ is not norm convergent to $d'_K(x)$. Then for every sequence $\{y_n\}$ in $E'(K)$ with $y_n \rightarrow x$, the sequence $\{f_{z_n}^{\rightarrow} - f_{y_n}^{\rightarrow}\}$ is not norm convergent to zero. So, there exists an $\epsilon > 0$ and a subsequence of $\{z_n\}$ (assume the sequence itself) such that for every sequence $\{y_n\}$ in $E'(K)$ with $y_n \rightarrow x$, we have

$$\|f_{z_n}^{\rightarrow} - f_{y_n}^{\rightarrow}\| > 5\epsilon \quad \text{for all } n.$$

So there exists a sequence $\{v_n\}$ in X with $\|v_n\| = 1$ such that

$$f_{y_n}^{\rightarrow}(v_n) - f_{z_n}^{\rightarrow}(v_n) > 5\epsilon \quad \text{for all } n.$$

Since d_K is Fréchet smooth at each $y_n \in E'(K)$, so there exists a $\delta_n(\epsilon, y_n) > 0$ such that for all $v \in X$ with $\|v\| = 1$, we have

$$\left| \frac{d_K(y_n + tv) - d_K(y_n)}{t} - f_{y_n}^{\rightarrow}(v) \right| < \epsilon \quad \text{for all } 0 < |t| \leq \delta_n.$$

So, for each n and $t_n > 0$ satisfying $\delta_n \geq t_n \downarrow 0$ we have

$$|d_K(y_n + t_n v_n) - d_K(y_n) - f_{y_n}^{\rightarrow}(t_n v_n)| < \epsilon t_n.$$

Put $t_n v_n = w_n$ so that $\|w_n\| = t_n$. So for each n , we have

$$\begin{aligned}
5\epsilon t_n &< f_{\vec{y}_n}(w_n) - f_{\vec{z}_n}(w_n) \\
&< f_{\vec{y}_n}(w_n) - d_K(y_n + w_n) + d_K(y_n) + d_K(z_n + w_n) - d_K(z_n) - f_{\vec{z}_n}(w_n) \\
&\quad + d_K(z_n) - d_K(y_n) + \|y_n - z_n\| \\
&< \epsilon t_n + \|z_n - p(z_n) + w_n\| - \|z_n - p(z_n)\| - f_{\vec{z}_n}(w_n) + 2\|y_n - z_n\|
\end{aligned}$$

Since norm is Fréchet smooth, in particular, at $z_n - p(z_n)$. So, there exists a $0 < \delta'_n(\epsilon, \vec{z}_n) < \delta_n$ such that for all $v \in X$ with $\|v\| = 1$, we have

$$\left| \frac{\|z_n - p(z_n) + tv\| - \|z_n - p(z_n)\|}{t} - f_{\vec{z}_n}(v) \right| < \epsilon \quad \text{for all } 0 < |t| \leq \delta'_n$$

So, for each n and t'_n satisfying $\delta'_n \geq t'_n \downarrow 0$ we have

$$\|z_n - p(z_n) + t'_n v_n\| - \|z_n - p(z_n)\| - f_{\vec{z}_n}(v_n) < \epsilon t'_n.$$

In particular, put $w_n = \delta'_n v_n$ then for all n , we have

$$\|z_n - p(z_n) + w_n\| - \|z_n - p(z_n)\| - f_{\vec{z}_n}(w_n) < \epsilon \delta'_n$$

Thus, we have

$$5\epsilon \delta'_n < \epsilon \delta'_n + \epsilon \delta'_n + 2\|y_n - z_n\|$$

This is true for every sequence $\{y_n\}$ in $E'(K)$ with $y_n \rightarrow x$, which is impossible. Since $E'(K)$ is dense in $X \setminus K$, for each n , we can choose $y_n \in E'(K)$ with $y_n \rightarrow x$ such that $\|y_n - z_n\| < \epsilon \delta'_n$. \square

Remark 2.3. In Theorem 4 of [4], author has shown that for an almost proximal set K , the set $E'(K)$ to be dense a sufficient condition on X is local uniform convexity (LUR) of the norm.

Let us denote by $E_r(K)$ the set

$$\left\{ x_o - r\vec{x}_o = x_o - r \frac{x_o - p(x_o)}{\|x_o - p(x_o)\|} : x_o \in E(K), p(x_o) \in P_K(x) \text{ and } \|x_o - p(x_o)\| = d_K(x_o) > r \right\}.$$

The following Theorem signifies that the uniform differentiability on a dense set will result in the differentiability on $X \setminus K$, if the norm on X is LUR and differentiable.

Theorem 2.1. Let X be a normed linear space with LUR and smooth (Fréchet smooth) norm. Let K be a nonempty closed and almost proximal set, if for some $r > 0$ the set $E_r(K)$ is dense in $X \setminus K$ and d_K is uniformly smooth (uniformly Fréchet smooth) on the dense set. Then distance function d_K is strictly smooth (and Fréchet smooth) on $X \setminus K$.

Proof. Let $\bar{x} \in X \setminus K$ and $\bar{r} > 0$ be arbitrary chosen. Then due to the denseness of $E_r(K)$ in $X \setminus K$, the set $E_r(K) \cap B(\bar{x}, \bar{r})$ is nonempty. Since d_K is uniformly smooth (uniformly Fréchet smooth) on $E_r(K) \cap B(\bar{x}, \bar{r})$. So, for given $\epsilon > 0$ and $y \in X$ with $\|y\| = 1$, there exists a $\delta(\epsilon, y) > 0$ ($\delta(\epsilon) > 0$) such that

$$\left| \frac{d_K(x + ty) - d_K(x)}{t} - f_{\bar{x}}(y) \right| < \epsilon \quad \text{for all } x \in E_r(K) \cap B(\bar{x}, \bar{r}), 0 < |t| < \delta.$$

So, for $x, z \in E_r(K) \cap B(\bar{x}, \bar{r})$ and for any $y \in X$ with $\|y\| = 1$, we have

$$\begin{aligned} |f_{\bar{z}}(y) - f_{\bar{x}}(y)| &\leq \left| \frac{d_K(z + ty) - d_K(z)}{t} - f_{\bar{z}}(y) \right| \\ &+ \left| \frac{d_K(z + ty) - d_K(z)}{t} - \frac{d_K(x + ty) - d_K(x)}{t} \right| \\ &+ \left| \frac{d_K(x + ty) - d_K(x)}{t} - f_{\bar{x}}(y) \right| \\ &< 2\epsilon + \frac{4}{\delta} \|z - x\| \quad \text{for all } \frac{\delta}{2} < |t| < \delta \\ &< 6\epsilon \quad \text{for all } \|z - x\| < \epsilon\delta. \end{aligned}$$

That is, the mapping $x \longrightarrow f_{\bar{x}}(y)$ ($x \longrightarrow f_{\bar{x}}$) is uniformly continuous on $E_r(K) \cap B(\bar{x}, \bar{r})$. Since $E_r(K)$ is dense in $X \setminus K$, this mapping has a unique continuous extension on $B(\bar{x}, \bar{r})$. But this implies that for any $x \in B(\bar{x}, \bar{r})$ and sequence $\{z_n\}$ in $E_r(K) \cap B(\bar{x}, \bar{r})$ converging to x , the sequence $\{f_{\bar{z}_n}\}$ is w^* -convergent (norm convergent). From Proposition 2.2, it follows that d_K is strictly smooth (and Fréchet smooth) at x . \square

One of the main results of this paper is to investigate the conditions on a nonempty closed set K such that $E_r(K)$ is dense in $X \setminus K$. A simple observation reveals that for almost sun K the set $E_r(K)$ is dense in $X \setminus K$. Indeed, we have the following result.

Lemma 2.3. *Let K be a nonempty closed and almost sun in a normed linear space X . Then for every $r > 0$, the set $E_r(K)$ is dense in $X \setminus K$.*

Proof. Let $\mathbb{S}(K)$ denotes the set of points $x \in X \setminus K$ where K is sun. Since K is an almost sun so the set $\mathbb{S}(K)$ is dense in $X \setminus K$, so it suffices to prove that $E_r(K)$ is dense in $\mathbb{S}(K)$.

Let $y \in \mathbb{S}(K)$ be any point. For all $0 < \epsilon < \|y - p(y)\|$ and $|t| < \epsilon$, if we choose

$$x_o = y + (r - t) \frac{y - p(y)}{\|y - p(y)\|}.$$

Then x_o is also in $\mathbb{S}(K)$ and it is easy to prove that $p(y) \in K$ is the nearest point for x_o and $d_K(x_o) = \|x_o - p(x_o)\| = \|x_o - p(y)\| = \|y - p(y)\| - t + r > r$. Now $\vec{x}_o = \vec{y}$, Therefore

$$\begin{aligned} x &= x_o - r\vec{x}_o \in \mathbb{E}_r(K) \\ &= x_o - r \frac{y - p(y)}{\|y - p(y)\|} \end{aligned}$$

$$\text{that is, } x = y - t \frac{y - p(y)}{\|y - p(y)\|},$$

$$\text{So, } \|x - y\| = |t| < \epsilon.$$

Thus, for $y \in \mathbb{S}(K)$ and $0 < \epsilon < \|y - p(y)\|$, the point $x = y - t\vec{y}$ is in $E_r(K)$ for all $|t| < \epsilon$ such that $\|x - y\| < \epsilon$. This proves that $E_r(K)$ is dense in $\mathbb{S}(K)$ for all $r > 0$. \square

Using Lemma 2.3 and Theorem 2.1, we can deduce the differentiability of d_K on $X \setminus K$, if it is uniformly differentiable on some dense set $E_r(K)$.

Corollary 2.1. *Let X be a normed linear space with LUR and smooth (Fréchet smooth) norm, and K be a nonempty closed subset of X , which is almost sun. Suppose d_K is uniformly smooth (uniformly Fréchet smooth) on the set $E_r(K)$ for some $r > 0$. Then distance function d_K is strictly smooth (and Fréchet smooth) on $X \setminus K$.*

3 Convexity of Almost Sun

We observe that the notion of almost sun is not merely a tidier form for Giles [6] page 462, but it also provides a non-trivial illustration for the proximal condition. Moreover, the Lemma 2.3 motivates to improve the results of Giles [6] page 462.

Theorem 3.1. *Let X be a normed linear space with uniformly smooth (uniformly Fréchet smooth) norm, and K be a nonempty closed subset of X , which is almost sun. Then d_K is smooth (Fréchet smooth) on $X \setminus K$.*

Proof. In the Giles proof, it requires only the denseness of $E_r(K)$ in $X \setminus K$ for some $r > 0$, which follows from Lemma 2.3 for almost sun K . \square

The above Theorem enables us to give a better characterizations for the convexity of a set. Which follows directly from Proposition 1.1 and Corollary 2.1.

Corollary 3.1. *Let X be a Banach space, and K be a nonempty closed subset of X , which is almost sun. If*

(i) X has uniformly smooth norm and the distance function d_K satisfies $\|d'_K(x)\| = 1$ for all $x \in X \setminus K$.

or (ii) X has LUR and smooth norm and the distance function d_K is uniformly smooth on the set $E_r(K)$ for some $r > 0$ which satisfies $\|d'_K(x)\| = 1$ for all $x \in X \setminus K$.

or (iii) X has uniformly Fréchet smooth norm,

or (iv) X has LUR and Fréchet smooth norm and the distance function d_K is uniformly Fréchet smooth on the set $E_r(K)$ for some $r > 0$.

Then K is convex.

Observe that if the distance function d_K generated by a proximal set K is smooth on $X \setminus K$, then we have $\|d'_K(x)\| = 1$ for all $x \in X \setminus K$. So if X has uniformly smooth norm, then every proximal and almost sun K is convex.

Let the norm of space X be uniformly smooth and rotund. Let K be an almost sun then every point $x \in X \setminus K$ which has a closest point in K has a unique closest point in K . To see this, suppose that a point $x \in X \setminus K$ has two closest points $p_1(x), p_2(x) \in K$. If \vec{x}_1 and \vec{x}_2 denotes the unit vectors in the direction of $x - p_1(x)$ and $x - p_2(x)$ respectively, then using Theorem 3.1, it follows that d_K is smooth at x and $d'_K(x) = f_{\vec{x}_1} = f_{\vec{x}_2}$. Since $f_{\vec{x}_1}(\frac{\vec{x}_1 + \vec{x}_2}{2}) = \left(\frac{f_{\vec{x}_1}(\vec{x}_1) + f_{\vec{x}_1}(\vec{x}_2)}{2} \right) = 1$, which implies that $\|\frac{\vec{x}_1 + \vec{x}_2}{2}\| = 1$, a contradiction to the rotundity. This proves the uniqueness of the closest point.

Thus we conclude the following result, which asserts that proximality is equivalent to Chebyshev property for almost sun.

Theorem 3.2. *Let X be a normed linear space with uniformly smooth and rotund norm. Then a nonempty closed set K which is almost sun, is Chebyshev if and only if K is proximal.*

Since every Hilbert space has the property of rotundity and uniform smoothness of the norm, and every Chebyshev set is proximal, Hence we have a partial result regarding the convexity of Chebyshev sets in a Hilbert space.

Theorem 3.3. *In a Hilbert space every Chebyshev set which is almost sun, is convex.*

Thus the problem of convexity of Chebyshev set in a Hilbert space is equivalent to the existence of a Chebyshev set K which is not a sun at every point of some open ball in $X \setminus K$.

It is known that in any reflexive Banach space X with Kadec norm, every nonempty closed set K has a set $\mathbb{E}(K)$ dense in $X \setminus K$, in particular every Hilbert space has this property. Thus, if for some $r > 0$ the set $\mathbb{E}_r(K)$ is dense in $X \setminus K$ then it follows that in a Hilbert space every Chebyshev set must be convex.

It is easy to verify that the Vlasov's differentiability condition is a consequence of the almost sun property and so we have the following result which is more general than the above Theorem.

Theorem 3.4. *In a Banach space X with rotund dual X^* every nonempty closed set K which is almost sun, is convex.*

Proof. Since K is almost sun, by Lemma 2.3 for every $r > 0$ the set $E_r(K)$ is dense in $X \setminus K$. So the proof follows from the ending Theorem of Giles[6] page 463. \square

Since a convex proximal set is a sun, so we have the following characterization and equivalent conditions for convexity of Chebyshev and proximal sets respectively.

Corollary 3.2. *Let X be a Banach space with rotund dual X^* . Then a Chebyshev set K is convex if and only if it is an almost sun.*

Theorem 3.5. *Let X be a Banach space with rotund dual X^* , and K be a proximal set in X . Then following are equivalent for K*

- (i) *Almost sun*
- (ii) *Convex*
- (iii) *Sun*

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